Patience Cycles

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Working Paper No. 09011
June 2009

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June 3, 2009

Abstract

There is a large body of evidence supporting the notion that a) those who grow up to be patient (forward-looking) do better in life compared to those who do not, and b) parents can inculcate the virtue of delayed gratification in their children by taking the right sort of actions. We study a dynamic model in which parents, for selfish reasons, invest resources to raise patient children. Patience raises the marginal return to human capital accumulation. The patient young do better in school, and hence, get more education but scrimp on investing in their own progeny's patience. This dynamic can generate intergenerational patience cycles. Generations coming of age with little patience will invest more in the productive capacity of their children, while those with greater patience invest more in their own productive capacity.

Keywords: patience, delayed gratification, human capital
JEL: E 2, J 24

*Puhakka thanks the OKO Bank and Yrjö Jahnsson Foundations for support. We thank the seminar audience at the University of Jyväskylä for comments.

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"Patience is the companion of wisdom."

— St. Augustine

1 Introduction

The ability to delay gratification is a highly desirable quality, one that is strongly correlated with success and income later in life. In a classic 1960s study (see Lehrer, 2009), researcher Michael Mischel offered a group of hungry four-year olds one marshmallow as an immediate reward, but, if they waited for him to return (after about 20 minutes), they could have two marshmallows instead. Fourteen years later, the longitudinal study demonstrated significant differences between these two groups. The children who delayed gratification and waited exhibited more positive and persistent traits as young adults; they were more self-motivated and were able to delay immediate gratification in order to pursue longer term goals. On the other hand, the children who chose one marshmallow exhibited traits of greater indecisiveness and mistrust of others later in life; they were less self-confident and often more troubled in general. Comparing SAT scores, Mischel found that students that chose 1 marshmallow scored an average of 210 points lower than the 2 marshmallow students. The overall message of this famed study is clear: those who grow up to be patient eventually do better in life.

To some extent, the ability to delay gratification has its roots in genetics, but mostly, it is a trait passed actively from parents to children. Parents routinely spend substantial resources in the form of time, effort, and energy to inculcate the virtues of waiting.[1] By teaching (or arranging to teach) a child how to play the violin or the piano, a parent is indirectly teaching the child the benefit of waiting for future rewards (such as, the accolades and attention following music performance). Similarly, by encouraging a child

[1] Several parenting-help websites offered such advice as: a) "When a 3-year-old asks for a cookie: "You can have one cookie now, or you can pick up your toys and have two cookies when you finish.", b) "When a 5-year-old wants to watch a favorite movie: "There isn't time to watch all of it now, but if you wait until after supper, we can watch it together and I'll make popcorn.""

to take on tough problems in arithmetic, a parent is teaching her to substitute, implicitly, the immediate gratification of playing with her dolls for the delayed gratification of joining Harvard some day.[2]

This desire to raise a patient child is partly motivated by altruistic feelings on the part of the parent. Parents want the best for their children, and if patience is a virtue, their children should have more of it. Presumably though, the parent also benefits directly from having a patient child.[3] Clearly, parents are happier if their child/adolescent is financially responsible, saves for his/her own future, delays sexual activity, and so on.

The child, as she comes of age, is presumably also happier and definitely more successful if raised to delay gratification. Bembenutty and Karabenick (2004) showed that academic success is largely dependent on students' ability to resist temptations of immediate gratification in order to increase the likelihood of accomplishing more important long-term goals. David Brooks (2006) describes the same in this fashion: "Young people who can delay gratification can sit through sometimes boring classes to get a degree. They can perform rote tasks in order to, say, master a language. They can avoid drugs and alcohol. For people without self-control skills, however, school is a series of failed ordeals. No wonder they drop out. Life is a parade of foolish decisions: teenage pregnancy, drug use, gambling, truancy and crime."

This paper incorporates the aforediscussed ideas into a simple reduced-form model.

[2]From Robert E. Lucas *Lectures on Economic Growth* (2001): "In my neighborhood in Chicago I bring my shirts to a laundry operated by a Korean woman, recently arrived, whose English is barely adequate to enable her to conduct her business. Her shop is open from 7 to 7, six days a week. As I enter, her 3-year-old daughter is seated on the counter being drilled in arithmetic—which she is very good at and clearly enjoys enormously. Fifteen years from now this girl will be beginning her studies at Chicago or Caltech, alongside the children of professors and Mayflower descendants."

[3]As one of the parenting-help websites argued "Teaching them how to wait for things they want could mean the difference between raising a child who successfully launches someday - and ends up living on his own, rather than back in his old bedroom - and one who does not."

Another website brought up the following example: "Rick and his wife, Corinne, found it difficult to go anywhere with the kids. If they weren't fighting in the car, the boys took off in three different directions the moment they were let loose in public. Rather than go on family outings and deal with Peter, Rhys, and Caleb's bolting, Rick and Corinne chose to stay home most of the time. They felt like prisoners in their own home."

Consider a dynamic economy in which agents spend valuable resources (invest in *patience capital*) to increase the discount factor of their children. Children take their own discount factor as given and decide how much (costly) education to acquire above and beyond a given baseline level. More education increases the amount of marketable human capital; ceteris paribus, the more patient you are, the higher is the contribution to human capital of an additional year of schooling. Higher human capital translates into higher future income.

The music lesson example provides a useful illustration of the type of broad effects that instilling patience in a child may have, which we attempt to model here. Patience can be viewed as part and parcel of learning a skill set – music lessons develop a child's concentration, abstract thinking, manual dexterity, and, along the way, the ability to delay gratification. Our treatment of patience captures this notion – namely, it enhances skill sets that individuals may acquire later in life *as well as* sets the stage for the child to grow up with a more forward (or patient) versus immediate outlook on life. These, we believe, are desirable traits for parents to nurture in their children.

A stripped-down version of this model is able to generate interesting generational cycles in patience. For example, consider a two-period cycle in which an agent starts off life with a high level of patience. This raises the return to getting an education, and she substitutes out of investing in her child's patience to pay for this extra education for herself. Her future income is higher but, now, her child starts off with low patience and has low income.

These cycles have potential to speak to a well-documented fact (Carliner, 1980; Borjas, 1993): earnings of second-generation workers in the U.S. (and elsewhere) are substantially higher than those of the first and the third. Existing work has suggested that it is the lack of opportunities for the first and the lack of motivation of the third that explain why the second generation does better. Instead, we argue it is the first generation that spends time and resources to teach the second generation to delay gratification (see footnote 2 above). This raises the return to schooling for the patient second generation. They choose to get a lot of education but scrimp on investing in patience for their own children; somewhat tongue-in-cheek, they end up raising the proverbial underachieving professor's kid.

The notion of endogenous discount rates goes back to Uzawa (1968). The idea that patience is an investment good was introduced in Becker and Mulligan (1997). In their setup, an individual at the start of life takes a rational decision to invest in patience, which in turn, determines the discount factor he uses for the rest of his life. In our formulation, which is much closer to that of Doepke and Zilibotti (2008), parents make the investment in their child's patience; subsequently, the child takes his own discount factor as given.[4] Another point of departure for us is that parental involvement in the preference formation of their child is not motivated by altruistic concerns. The current paper is also thematically connected to Akabayashi (2006) and Bhatt and Ogaki (2008), although the focus of our paper differs sharply from theirs. In Akabayashi (2006), a child's discount factor depends on her human capital which, among other things, depends on how much time the parent spends with the child. In Bhatt and Ogaki (2008), a young agent's discount factor depends on her consumption as a small child, and altruistic parents can manipulate their child's discount factor by holding back on transfers ("tough love").

The remainder of this paper is as follows. Section 2 lays out the general model environment, including some basic properties on agents' expenditures on education and patience capital. In Section 3, we discuss the equilibrium properties of the model. Here we establish conditions which ensure a unique steady state obtains. We also explore the possibility that the model can support 2-period equilibrium limit cycles and provide a few numerical examples of cycle equilibria. Section 4 concludes.

2 The model

2.1 Environment

We consider a simple and highly-stylized model economy populated by endless cohorts of three-period lived overlapping generations. Label these three periods of life as *childhood*,

[4]See Skog (2001) for a critique of the Becker-Mulligan (1997) idea.

youth, and *old-age*. Let $t = 1, 2, ..., \infty$ denote time.

A young agent is endowed with y units of a perishable consumption good, and can work when old. There are no saving instruments. When young, an agent has exactly one offspring. The child makes no decisions and does not care about consumption.

In addition to caring about consumption during youth and old age, the agent receives utility (*warm glow*) from raising a patient child. Specifically, let c_t (x_t) denote consumption when young (old) for an agent born at date $t-1$, and let β_t denote the discount factor between t and $t+1$. Then, her utility function is given by

$$U\left(c_t, x_t, \beta_{t+1}; \beta_t\right) = u\left(c_t\right) + \beta_t\, u\left(x_t\right) + \theta\, v\left(\beta_{t+1}\right),$$

where $\theta > 0$ is a parameter; additional structure on the functions u and v will be imposed below. A young agent at date t takes β_t as given but cares about influencing the β of her child – captured by the function $v(\cdot)$. A child raised with a high β is more forward-looking (can delay gratification) compared to one raised with low β. A young parent invests an amount p_t (henceforth "patience capital") towards the raising of a more patient child. Aside from an innate endowment of some patience capital, we assume that the child's β is entirely shaped by the amount spent by her parents; in particular, $\beta_{t+1} = g\left(p_t\right)$, where $g(0) > 0$. Additional assumptions on the function $g(\cdot)$ are made below.

The idea that parents shape the discount factor of a child is similar in spirit to that of Becker and Mulligan (1997). In our setup, however, parents are not altruistic towards their children. They simply receive a warm glow from raising a "more patient" child; specifically, they are assumed to get direct utility from raising a child whose consumption is more tilted towards old age than youth. In effect, the parent is spending resources to produce a calmer, less-fidgety child who will delay gratification. Examples of such investments include teaching children to wait their turn, taking them to music lessons (where the reward of playing in front of an audience comes only after prolonged practice and perseverance), teaching them chess (a game that requires patience), inculcating good reading habits (encouraging them to look forward to the dénouement of the story's plot), and so on. The parent reaps a direct benefit of having such a child, presumably in the form of more pleasant restaurant visits, better grades in school, less need of disciplining, less behavioral problems, and the like. Below, we model how the child benefits from being raised patient.

We think of the agent's young-age endowment, y, as the goods-equivalent of basic skills or human capital that she acquires with relatively little effort or investment (say, education up to high school). An agent may use part of this endowment to acquire additional 'marketable' human capital – call it higher education – achieved by incurring educational expenses, e_t, at a unit cost ϕ. We assume expenses on education gets converted into marketable human capital next period via a production function:

$$h_{t+1} = h\left(e_t; \beta_t\right).$$

An old agent with human capital h_{t+1} earns income $w\, h_{t+1}$, where $w > 0$ is a parameter.

The idea behind the formulation is simple: patient children do better in school. A patient student generates more human capital (for the same education expense) when compared to an impatient student. Here lies the benefit to a child from being raised to delay gratification; such patience is rewarded by the education system (in the form of more human capital accumulated), and indirectly by the marketplace.

We collect all the technical assumptions on the functions $u, v, g,$and h below. These are maintained in all that follows.

Assumptions

A1 $u\left(\cdot\right)$ *is strictly concave, with* $\lim_{c\to 0} u'\left(c\right) = \infty$.

A2 $v\left(\cdot\right)$ *is concave.*

A3 $g\left(\cdot\right)$ *is concave, with* $g\left(0\right) > 0$.

A4 $h_1 > 0$, $h_{11} < 0$, $h_{12} \geq 0$, and $h_2 \geq 0$ *with* $\lim_{e\to 0} h_1\left(e;\beta\right) = \infty$ for $\beta > 0$.

Assumption A3 posits that agents are born with some innate patience. We include some additional assumptions on these functions in the analysis below. In passing, note that since a parent determines her child's β, and this choice enters the parent's utility only through a warm-glow effect, our framework is general enough to permit scale economies in the production of human capital.

2.2 The agent's problem

Given β_t, a young agent's decision problem at date t is as follows:

$$\max_{c_t, x_t, \beta_{t+1}} \quad u(c_t) + \beta_t u(x_t) + \theta v\left(\beta_{t+1}\right)$$

subject to

C1 $c_t + \phi e_t + p_t \leq y$

C2 $x_t \leq w\ h_{t+1}$

C3 $h_{t+1} = h(e_t; \beta_t)\,; \quad \beta_{t+1} = g(p_t)$

C4 $c_t \geq 0,\ x_t \geq 0,\ p_t \geq 0$

Assuming interior solutions for all choice variables, the first-order conditions for the agent's problem are

F1 $c_t: \quad u'(c_t) = \lambda_{yt}$

F2 $x_t: \quad \beta_t\, u'(x_t) = \lambda_{ot}$

F3 $p_t: \quad \theta\, v'\left(\beta_{t+1}\right)\, g'(p_t) = \lambda_{yt}$

F4 $e_t: \quad \phi\lambda_{yt} = w\lambda_{ot}\, h_1(e_t; \beta_t)$

where λ_{it} is the Lagrangian multiplier on the resource constraint of an agent in her i^{th} stage of life.

Using F1 and F2, F4 can be written as:

$$\phi\, u'(c_t) = w\beta_t\, u'(x_t)\ h_1(e_t;\beta_t)\,. \tag{1}$$

Using F1, F3 can be written as:

$$\theta\, v'\left(\beta_{t+1}\right)\ g'(p_t) = u'(c_t)\,. \tag{2}$$

Parenthetically, (1) equates the marginal cost of education (in terms of forgone utility, $\phi u'(c_t)$) with the marginal benefit of education: higher future wage earnings $w\ h_1(e_t;\beta_t)$, which, in terms of future utility, is $w\beta_t u'(x_t)\, h_1(e_t;\beta_t)$.

Note that a parental choice of investment in patience affects the right hand side of her child's marginal condition (1) in two ways – first, directly, through its effect on her discount factor β_t – how she views future consumption relative to current consumption – and second, through the effect patience has on the human capital production and the effectiveness of a marginal investment in education, $h_1(e_t;\beta_t)$. This latter effect is one of the novelties of our framework. Our treatment recognizes the fact that instilling greater patience in one's child may do more than simply shaping the child's discount factor β – it can, in fact, enhance the child's own cognitive abilities, raising, as we note, the marginal benefits of education.

Using the budget constraints and the technologies, we can rewrite (1) and (2) as:

$$\phi u'(y - \phi e_t - p_t) = wg(p_{t-1})\, u'(wh(e_t; g(p_{t-1})))\, h_1(e_t; g(p_{t-1}))\,. \tag{3}$$

$$\theta v'(g(p_t))\, g'(p_t) = u'(y - \phi e_t - p_t)\,. \tag{4}$$

This last equation implicitly defines a function, $e_t = e(p_t)$. The following lemma summarizes some results concerning the function $e(p_t)$.

Lemma 1 *Suppose* $\lim_{p\to 0} v'(g(p))\, g'(p) = \infty$ *and that Assumptions A1-A4 hold. Then,*
i. $\phi e'(p_t) \leq -1$.

ii. There exists a y^*, $y^* < y$ *such that* $\lim_{p \to y^*} \phi e(p) = 0$, *and*

iii. $\lim_{p \to 0} \phi e(p) = y$.

Part (i) of Lemma 1 indicates two things. First, it says that a parent's expenditure on her own education is a decreasing function of her choice of patience capital for her child. The result is not too surprising, considering investment in one's own education competes directly for resources with patience capital in the agent's budget constraint, C1. Second, it states that total expenditures, $p_t + \phi e(p_t)$, are *decreasing in* p_t. This suggests that, *ceteris paribus*, a unit increase in spending on patience will result in a proportionately larger reduction in expenditures on education – the difference is spent on first-period consumption. Part (ii) follows directly from (4) and the fact that the derivative $v'(g(p))\, g'(p)$ is bounded at $p = y$, while $u'(y - \phi e_t - p_t)$ is unbounded if $\lim_{p \to y} \phi e(p) + p = y$, i.e., if $\lim_{p \to y} \phi e(p) = 0$. Part (iii) follows from the fact that $\lim_{p \to 0} v'(g(p))\, g'(p) = \infty$.

A proof of Lemma 1 and of all other propositions to follow may be found in the Appendix.

3 Equilibria

3.1 The law of motion and steady-state

Using $e(p_t)$, (3) implicitly defines a law-of-motion for patience capital, $p_t = \mathcal{P}(p_{t-1})$:

$$\phi u'(y - \phi e(p_t) - p_t) = wg(p_{t-1})\, u'(wh(e(p_t); g(p_{t-1})))\, h_1(e(p_t); g(p_{t-1})). \tag{5}$$

It serves to connect an agent's discount rate with that of her child's.

Given an initial stock of patience, p_0, an equilibrium can be summarized by a sequence $\{p_t\}_{t=0}^{\infty}$, $0 \leq p_t < y$ that satisfies (5), where $e(p_t)$ is implicitly defined by (4). Below, we establish conditions that ensure the existence of a steady state equilibrium. We will turn to the issue of non-stationary equilibria in Section 3.2.

A steady state p^* satisfies

$$\phi u'(y - \phi e(p^*) - p^*) = wg(p^*)\, u'(wh(e(p^*); g(p^*)))\, h_1(e(p^*); g(p^*)). \tag{6}$$

Proposition 1 *There exists a steady-state, $p^* \in (0, y^*)$. If, in addition, $h_2(e,p) = 0$, the steady state is unique.*

Interestingly, it is easy to establish uniqueness of the steady state if h is independent of β; of course, the latter is a sufficient condition.

We are now in a position to investigate further the dynamics of patience p_t. Differentiating both sides of (5), it is easy to show that

$$\mathcal{P}'(p_{t-1}) \equiv \frac{dp_t}{dp_{t-1}} = -\frac{\mathcal{N}}{\mathcal{D}},$$

where

$$\mathcal{N} \equiv [w^2 g(p_{t-1}) u''(x_t) h_1(e(p_t); g(p_{t-1})) h_2(e(p_t); g(p_{t-1})) \\ + wg(p_{t-1}) u'(x_t) h_{12}(e(p_t); g(p_{t-1})) + wu'(x_t) h_1(e_t; g(p_{t-1}))] \, g'(p_{t-1})$$

and

$$\mathcal{D} \equiv \phi u''(c_t) \left(\phi e'(p_t) + 1\right) + w^2 g(p_{t-1}) u''(x_t) \left(h_1(e(p_t); g(p_{t-1}))\right)^2 e'(p_t) \\ + wg(p_{t-1}) u'(x_t) h_{11}(e(p_t); g(p_{t-1})) e'(p_t) > 0.$$

with $c_t = y - \phi e_t - p_t$ and $x_t = w\, h(e_t; g(p_{t-1}))$.

Proposition 2 *If $h_2(e, \beta) = 0$, i.e., if patience does not influence the accumulation of human capital,*

$$\frac{dp_t}{dp_{t-1}} \leq 0$$

implying the law of motion for p is non-increasing everywhere.

Clearly, $\mathcal{D} > 0$ follows from the concavity properties of $u(\cdot)$ and Part (i) of Lemma 1 – the fact that $e'(p_t) < 0$ and $\phi e'(p_t) + 1 \leq 0$ over the domain of p. In general, the sign of $\mathcal{N}$ is not conclusive – the first term inside the square brackets is negative; if $h_{12} > 0$, the other two terms are positive. In the special case where $h_2(e(p_t); g(p_{t-1})) = 0$, $\mathcal{N}$ is positive, and the law-of-motion for p is downward sloped over the entire domain of p. In this case, combining the insight from Proposition 1, we know that there is a unique steady state and the slope at the steady state is negative.

What happens when $h_{12} > 0$? The following additional assumptions impose more structure on preferences and technology and help sign the slope of the law of motion for p.

Assumptions

A5 $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, $\sigma > 0$, $\sigma \neq 1$; $u(c) = \ln c$, when $\sigma = 1$

A6 $h(e_t; \beta_t) = Ae_t^{\mu}\beta_t^{\eta}$, with $A > 0$, $0 \leq \mu \leq 1$ and $\eta \geq 0$.

Using A5-A6, the budget constraint, $x_t = w\, Ae_t^{\mu}\beta_t^{\eta}$ and the fact that $wg(p_{t-1})\, h_2(e(p_t); g(p_{t-1})) = \eta w A e_t^{\mu}\beta_t^{\eta} = \eta x_t$, it is possible to compute $\mathcal{N}$ as

$$\mathcal{N} = (-\sigma\eta + 1 + \eta)(x_t)^{-\sigma}\, w\mu\, Ae_t^{\mu-1}\beta_t^{\eta} g'(p_{t-1}).$$

Corollary 1 *Suppose A5-A6 hold. Then a necessary and sufficient condition for* $\frac{dp_t}{dp_{t-1}} \leq 0$ *is* $1 + \eta(1-\sigma) > 0$.

Note that if A5-A6 hold, the sign of $\frac{dp_t}{dp_{t-1}}$ is either positive or negative depending on $1 + \eta(1-\sigma) \lesseqqgtr 0$, i.e., the time-map $\mathcal{P}$ cannot exhibit any sort of non-monotonicity. It deserves mention here that $\sigma \leq 1$ is necessary to obtain $\frac{dp_t}{dp_{t-1}} \leq 0$.

Assuming A5-A6 hold, we can write the two conditions necessary for an interior optimum as:

$$\phi(y - \phi e_t - p_t)^{-\sigma} = g(p_{t-1})\left(wAe_t^{\mu} g(p_{t-1})^{\eta}\right)^{-\sigma} w\mu A e_t^{\mu-1} g(p_{t-1})^{\eta} \tag{7}$$

and

$$\theta\nu' g'(p_t) = (y - \phi e_t - p_t)^{-\sigma},$$

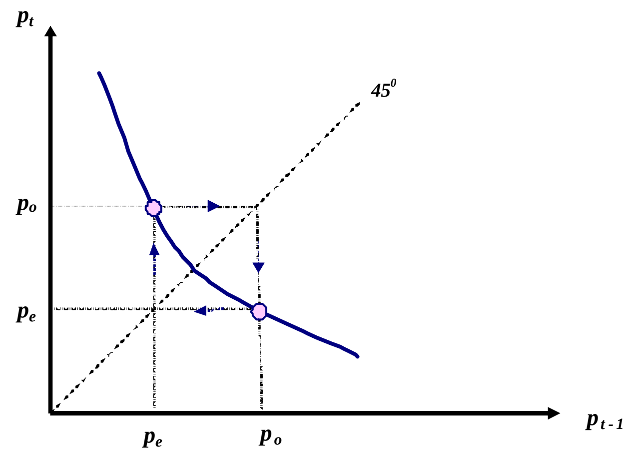

Figure 1: Two-Period Limit Cycles in Patience Capital

where, for ease of presentation, we define $\nu' \equiv \nu'(g(p_t))$. It is easy to verify that the equilibrium law of motion for p is implicitly given by

$$p_t = y - \phi \left(\frac{\mu g(p_{t-1})^{1+\eta(1-\sigma)} (wA)^{1-\sigma}}{\phi\theta\nu' g'(p_t)} \right)^{\frac{1}{1-\mu(1-\sigma)}} - \left(\theta\nu' g'(p_t)\right)^{-\frac{1}{\sigma}}, \tag{8}$$

and the expression for education expenses is given by

$$e_t = \left(\frac{\mu g(p_{t-1}) (wAg(p_{t-1})^{\eta})^{1-\sigma}}{\phi\theta\nu' g'(p_t)} \right)^{\frac{1}{1-\mu(1-\sigma)}}. \tag{9}$$

3.2 Two-period limit cycles

A technical implication of a law of motion that is decreasing monotonically is that cycles of periodicity greater than two are not possible. (For that, a necessary condition is the law of motion should be non-monotonic) Next, we consider the possibility of generating two-period limit cycles. Such cycles are pairs p_o and p_e ($p_o \neq p_e$) such that $p_t = p_o$ (at an odd date) and $p_t = p_e$ (at an even date) for all t. Figure 1 depicts such a cycle equilibrium.

A sufficient condition to obtain two-period limit cycles in a neighborhood of the steady state, p^*, is

$$\frac{dp_t}{dp_{t-1}}|_{p^*} = -1.$$

Below, we investigate the implications of this condition in the context of an analytically transparent example.

From here on, assume $\nu(\cdot)$ and $g(\cdot)$ are linear functions, and that $\sigma = 1$ [$u(\cdot)$ is the log function]. [5]Then using (8), we can get

$$p_t = \Lambda - \Gamma\, g(p_{t-1}) \tag{10}$$

where $\Lambda \equiv y - \frac{1}{\theta\nu' g'}$, and $\Gamma \equiv \frac{\mu}{\theta\nu' g'} = -\mu\Lambda + \mu y. > 0$. It follows from Corollary 1 that $\frac{dp_t}{dp_{t-1}} \leq 0$.

Two-period cycle equilibria exist if $\Gamma g' = 1$; i.e., if $\mu = \theta\nu'$. Assuming $\mu = \theta\nu'$, the right-hand side of (10) maps the interval $[0, \Lambda - \Gamma g(0)]$ into itself, with a fixed point $p^* = \frac{\Lambda - \Gamma g(0)}{2}$. In order to ensure $0 \leq p_t \leq y$ holds and the discount factor, β_t, satisfies $0 \leq \beta_t < 1$, we require: i) $0 < \Lambda - \Gamma g(0) < y$, and ii) $g(\Lambda - \Gamma g(0)) < 1$. Clearly, $\Lambda - \Gamma g(0) < y$ is satisfied for all increasing functions $\nu(\cdot)$ and $g(\cdot)$ with $g(0) \geq 0$, since $\Lambda - \Gamma g(0) = y - \frac{1 + \mu g(0)}{\theta\nu' g'}$. Let

$$g(p) = \varphi + \pi p, \ \varphi > 0, \pi > 0.$$

Then $g'(p) = \pi$ and $g''(p) = 0$; the condition $0 < \Lambda - \Gamma g(0)$ is assured if $\varphi\mu + 1 < \mu\pi y$, given the assumption that $\mu = \theta\nu'$. In order for $\beta_t \leq 1$ over the entire domain $[0, \Lambda - \Gamma g(0)]$, we require $\pi\Lambda < 1$, i.e., $\pi\mu y < (1 + \mu)$.[6]

[5]The assumption here that $\nu(\cdot)$ and $g(\cdot)$ are linear functions does not satisfy the all the conditions of A1-A6 and in Lemma 1; the latter provide sufficient conditions to ensure the existence of a steady-state.

[6]Since the law of motion is downward sloped, p_t is greatest when $p_{t-1} = 0$. At that point, $p = \Lambda - \Gamma g(0) = \Lambda - \Gamma\varphi$, or twice the steady-state value p^*. Evaluating g at this point, we have $g(\Lambda - \Gamma\varphi) = \varphi + \pi(\Lambda - \varphi\Gamma) = \pi\Lambda$, since $\pi\Gamma = 1$.

Note that the condition $\pi\Lambda < 1$ is sufficient to ensure $\beta_t \leq 1$ for all t. If this condition is not met, one can restrict the analysis to a subset of the domain $[0, \Lambda - \Gamma g(0)]$ that ensures $\beta_t \leq 1$. That set is

A two-period cycle, in this instance, is characterized by a pair (p_o, p_e) that satisfies the following:

$$\begin{aligned} p_o &= \Lambda - \Gamma g(p_e) \\ p_e &= \Lambda - \Gamma g(p_o) \end{aligned} \tag{11}$$

Clearly, a pair with $p_o = p_e$ obtains at the steady state, p^*. The following proposition states that a two-cycle exists.

Proposition 3 *For any $p_o \in (0, \Lambda - \Gamma g(0))$, other than the steady state, p^*, there exists a $p_e \neq p_o$ such that (p_o, p_e) satisfies (11).*

The cycle in patience capital investment naturally induces generational cycles in other important macro aggregates, such as education expenditures, output, and even growth rates. To see this, notice that expenditures on education satisfy, $\phi e_t = \Gamma g(p_{t-1})$ implying

$$e_e = \frac{\Gamma}{\phi} g(p_o); \quad e_o = \frac{\Gamma}{\phi} g(p_e)$$

where the subscript e and o refer to even and odd dates, respectively. Similarly, noting that output at date t is defined as $Y_t \equiv h\left(e_{t-1}; \beta_{t-1}\right)$, we then have

$$Y_e = A\left(\frac{\Gamma}{\phi}\right)^{\mu} g(p_e)^{\mu+\eta}; \quad Y_o = A\left(\frac{\Gamma}{\phi}\right)^{\mu} g(p_o)^{\mu+\eta}$$

and from these, (gross) rates of growth γ are computed as

$$\gamma_o = \left(\frac{g(p_e)}{g(p_o)}\right)^{\mu+\eta}; \quad \gamma_e = \frac{1}{\gamma_o}.$$

The above computations highlight some of the more subtle aspects of patience in our model. Consider, for the sake of argument, that a two-cycle equilibrium exists with $p_e < p_o$.

$(\pi\Lambda - 1)/\pi, (1-\varphi)\Gamma$, which is obtained by finding the minimum value of p such that $g(\Lambda - \Gamma(\varphi + \pi p)) = 1$. That value is $(\pi\Lambda - 1)/\pi$. The upper limit of the interval $(1-\varphi)\Gamma$ is then obtained by evaluating the right-hand side of (10) at $(\pi\Lambda - 1)/\pi$.

This means the choice of patience capital of a young parent in odd dates is greater than that of a parent at even dates (so a generation born in odd dates will be more patient than those born in even dates). These odd-generation children, in turn, find each unit of investment in their own education to be more productive; hence, they spend fewer resources to instill patience in *their* children. Due to the timing of production in this model, output will be high in odd dates and low in even dates (so the growth rate in even dates will be greater than one in even dates and less than one in odd dates). In short, generations coming of age with less patience capital invest more in the productive capacity of their children, while those with greater patience find it prudent to invest more in their own productive capacity.

The following provides a numerical example of such cycle equilibria:

Example 1 *Let* $g(p) = \varphi + \pi p$; $v(\beta) = \beta$; $\pi = 0.25$; $\varphi = 0.16$; $\mu = 1$; $\eta = 1$; $\theta = 1$; $\sigma = 1$; $w = 2$; $A = 1$; $y = 8$.

A steady state for this economy obtains with a value for patience capital $p^* = 1.68$ and discount factor $\beta^* = 0.58$. A two-period cycle equilibrium also obtains, with $p_e = 0.73$ and $p_o = 2.63$; that is, with $\beta_e = 0.3425$, and $\beta_o = 0.8175$, respectively. Additionally, as shown above, a two-period cycle equilibrium exists for *any* initial $p \neq p^*$ in the interval $(0, \Lambda - \Gamma g(0)) = (0, 3.36)$.

Our discussion above imposes a number of restrictions on the model's primitives, perhaps leaving the impression such two-period cycles rest on a strict knife-edge case, $\Gamma g' = 1$ – a sufficient condition. Of course, cycles obtain under far less restrictive conditions, as the following example illustrates.

Example 2 *Let* $g(p) = \varphi + \pi p^{\alpha}$; $v(\beta) = \beta$; $\alpha = 0.8$; $\pi = 0.48$; $\varphi = 0.16$; $\mu = 1$; $\eta = 1$; $\theta = 0.9$; $\sigma = 0.9$; $w = 2$; $A = 1$; $y = 9.5$.

In this example, $p_e = 1.3058$, $p_o = 1.9719$, $\beta_e = 0.7542$, and $\beta_o = 0.9863$, with a steady state $p^* = 1.6234$ and $\beta^* = 0.8673$. Here, the slope at the steady state is in a neighborhood of -1, i.e., -1.0046. The growth rate, $\gamma_t = h_{t+1}/h_t$, in the steady state is 1, while in the cycle equilibrium $\gamma_e = 1.6564$ and $\gamma_o = 0.6037$. Recall, p_{t-1} is the amount of patience capital a young parent at date t receives from her parents; those born on odd dates will have higher effective human capital and are capable of producing a lot when old (also an odd date), hence the transition from odd to even dates exhibits a low (negative) rate of net growth.

4 Conclusions

Patience, as Rousseau notes, is bitter, but its fruit is sweet. This paper explores a simple model in which the sacrifices and rewards for cultivating patience are intergenerational in nature. The basic tenets of the model rest on two premises which, we feel, are fairly non-controversial: a) patience, like most any other factor, requires a commitment of real resources to acquire, and b) patience enhances human capital and the ability of individuals to acquire it.

In our set-up, parents commit resources to instill patience in their children. This, in turn, has two effects, both of which promote human capital acquisition. A forward-looking child is more keen to make future investment (as an adult) in human capital rather than increase current consumption. Secondly, by raising the marginal return to education, patience enhances the investment in human capital and offers the adult further incentive to invest more in learning. Interestingly, these two effects can produce intergenerational equilibrium cycles in patience, as the adult chooses to enhance her own human capital at the expense of investing in greater patience capital for her child. Such cycle equilibria possess some potential to explain why parents with little human capital sometimes produce high human capital progeny, who in turn, have children that exhibit lackluster talents. Loosely speaking, such equilibria can also address why second-generation immigrants often earn substantially more than those of the first and the third generations in their lineage.

To keep the analysis manageable, we assumed parents receive a warm glow from investment in their child's patience. This assumption could easily be replaced by altruistic preferences. Additionally, the model assumes that investment in a child's patience does not 'bear fruit' until much later in life. Both are open to more critical scrutiny – however, we conjecture that alternative formulations are likely to *increase* the complexity of the intergenerational patience dynamics, rather than reduce it.

Appendix

Proof of Lemma 1

Part (i): Differentiate both sides of (4) to get

$$\left(\theta v''\left(g\left(p_t\right)\right)\left(g'\left(p_t\right)\right)^2+\theta v'\left(g\left(p_t\right)\right)g''\left(p_t\right)\right)dp_t=-\phi u''\left(y-\phi e_t-p_t\right)de_t-u''\left(y-\phi e_t-p_t\right)dp_t.$$

Solving, we get:

$$e'\left(p_t\right)=-\frac{\left(\theta v''\left(g\left(p_t\right)\right)\left(g'\left(p_t\right)\right)^2+\theta v'\left(g\left(p_t\right)\right)g''\left(p_t\right)\right)+u''\left(y-\phi e_t-p_t\right)}{\phi u''\left(y-\phi e_t-p_t\right)}<0$$

Since

$$\phi e'\left(p_t\right)=-\frac{\left(\theta v''\left(g\left(p_t\right)\right)\left(g'\left(p_t\right)\right)^2+\theta v'\left(g\left(p_t\right)\right)g''\left(p_t\right)\right)}{u''\left(y-\phi e_t-p_t\right)}-1,$$

Part (ii) follows immediately, since

$$\frac{\partial}{\partial p_t}\left(p_t+\phi e\left(p_t\right)\right)=1+\phi e'\left(p_t\right)$$
$$=-\frac{\left(\theta v''\left(g\left(p_t\right)\right)\left(g'\left(p_t\right)\right)^2+\theta v'\left(g\left(p_t\right)\right)g''\left(p_t\right)\right)}{u''\left(y-\phi e_t-p_t\right)}<0$$

where the latter inequality follows from the concavity assumptions on $v(\cdot)$ and $g(\cdot)$.

Proofs of other parts of the lemma are provided in the main text. ■

Proof of Proposition 1

Let $LHS\left(p\right)\equiv\phi u'\left(y-\phi e\left(p\right)-p\right)$ and $RHS\equiv wg\left(p\right)u'\left(wh\left(e\left(p\right);g\left(p\right)\right)\right)h_1\left(e\left(p\right);g\left(p\right)\right).$ Differentiating LHS, we have:

$$LHS'\left(p\right)=-\phi u''\left(y-\phi e\left(p^*\right)-p^*\right)\left(\phi e'\left(p\right)+1\right)\leq 0,$$

with strict inequality over $(0,y^*)$, using Part (i) of Lemma 1, and the concavity of $u(\cdot)$. From Part (ii) of Lemma 1, $\lim_{p\to y^*}LHS\left(p\right)>0$. From Part (iii) of Lemma 1, $\lim_{p\to 0}LHS\left(p\right)=\infty$.

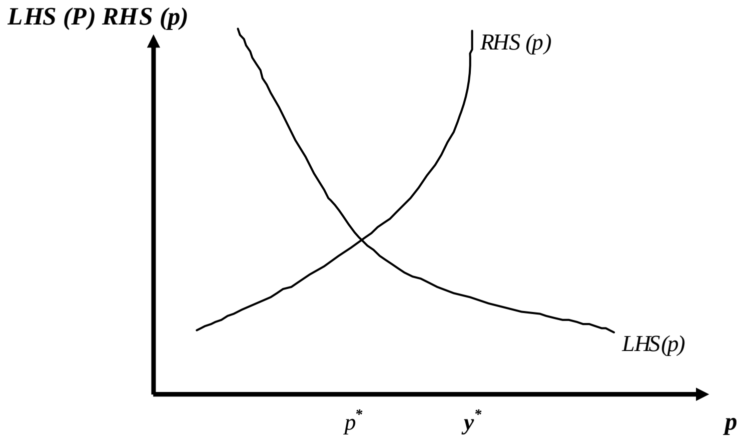

Figure 2: Steady-State in Patience Capital

Differentiating RHS, we have

$$
\begin{aligned}
RHS'(p) &= wu'(wh(e(p);g(p)))\,h_1(e(p);g(p))\,g'(p)\\
&+w^2 g(p)\,u''(wh(e(p);g(p)))\,(h_1(e(p);g(p)))^2\,e'(p)\\
&+w^2 g(p)\,u''(wh(e(p);g(p)))\,h_1(e(p);g(p))\,h_2(e(p);g(p))\,g'(p)\\
&+wg(p)\,u'(wh(e(p);g(p)))\,h_{11}(e(p);g(p))\,e'(p)\\
&+wg(p)\,u'(wh(e(p);g(p)))\,h_{12}(e(p);g(p))\,g'(p).
\end{aligned}
$$

The term above is positive everywhere if $h(e(p);g(p))$ is nonincreasing in p; that is, if $h_1(e(p);g(p))\,e'(p)+h_2(e(p);g(p))\,g'(p) \leq 0$. This holds, trivially, if $h_2 \equiv 0$. In this case, there exists a unique steady state in the interval $(0, y^*)$, since $\lim_{p \to y^*} wg(p)\,u'(wh(e(p);g(p)))\,h_1(e(p);g(p)) \to \infty$ as $e(p) \to 0$ (see Figure 2).

More generally, $RHS'(p)$ may not be increasing over the entire domain of p, and especially in a neighborhood of $p = 0$ if $\lim_{p\to 0} h_2(e(p);g(p))\,g'(p) = \infty$, i.e., $g(p)$ has the limit condition $\lim_{p\to 0} g'(p) = \infty$. However, since $g(0) > 0$ [see A3] we have $\lim_{p\to 0} RHS(p) < \infty$. Additionally, $\lim_{p\to y^*} RHS(p) = \infty$ follows from A4 and Part (ii) of Lemma 1. ■

Proof of Proposition 3

Given p_o, $p_e = \Lambda - \Gamma g\left(p_o\right)$. We need to show $p_o = \Lambda - \Gamma g\left(p_e\right)$. We have:

$$\begin{aligned}
\Lambda - \Gamma g\left(p_e\right) &= \Lambda - \Gamma\left(\varphi + \pi\left(\Lambda - \Gamma g\left(p_o\right)\right)\right) \\
&= \Lambda - \Gamma\varphi - \pi\Gamma\Lambda + \left(\pi\Gamma\right)\Gamma g\left(p_o\right) \\
&= \Lambda - \Gamma\varphi - \pi\Gamma\Lambda + \left(\pi\Gamma\right)\Gamma\left(\varphi + \pi p_o\right) \\
&= \Lambda - \Gamma\varphi - \pi\Gamma\Lambda + \left(\pi\Gamma\right)\Gamma\varphi + \left(\pi\Gamma\right)^2 p_o \\
&= p_o
\end{aligned}$$

since, by assumption, the slope of the law of motion is -1, or equivalently, $g'\Gamma = 1$, i.e., $\pi\Gamma = 1$.■

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